

On Hall's Counterexample

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Abstract

Modifying Hall's idea in [6] we construct an example of homeomorphism of the circle which is a Denjoy counterexample (*i.e.* it is not conjugated to a rotation) and which is a C^∞ -diffeomorphism everywhere except in a flat half-critical point.

1 Introduction

1.1 Notations and Definitions

Rotations. Let us denote by \mathbb{S}^1 the unit circle, for each $\rho \in \mathbb{R}$ let $R_\rho : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the map defined by

$$R_\rho(\theta) = (\theta + \rho) \mod \mathbb{Z}$$

which is called the *rotation of angle* ρ .

Lifts. Let $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the projection of the real line to the circle. Then, for each continue function $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ we can define a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\pi \circ F = f \circ \pi.$$

We call F a lift of f . In particular there exists a unique lift F of f such that $F(0) \in [0, 1)$.

Consider $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and a lift $F : \mathbb{R} \longrightarrow \mathbb{R}$ of f , then f and F have the same properties of regularity: continuity, differentiability, smoothness, *etc.*

Definition 1.1. Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a continue function and $F : \mathbb{R} \longrightarrow \mathbb{R}$ a lift of f . We say that f has degree one if for all $x \in \mathbb{R}$, $F(x + 1) = F(x) + 1$.

Definition 1.2. A function $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is non-decreasing¹ if one of its lift is non-decreasing (as a function from the reals to the reals).

¹or orientation preserving

1.2 Rotation number

Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a non-decreasing, degree one, continue function and let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a lift of f . Then the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} (F^n(x) - x)$$

exists for every $x \in \mathbb{R}$ and it is independent of x . We denote it by $\tilde{\rho}(F)$.

Notice that this quantity does not depend on the choice of the lift: if \tilde{F} is another lift of f , then $\tilde{\rho}(F) - \tilde{\rho}(\tilde{F}) = F - \tilde{F} \in \mathbb{Z}$. This leads to the following:

Definition 1.3. Let F be a lift of f . The rotation number $\rho(f)$ of f is defined as the class of $\tilde{\rho}(F)$ modulo 1.

Proposition 1. $\rho(\cdot)$ is non-decreasing: if $f_1 \prec f_2$, then $\rho(f_1) \leq \rho(f_2)$.

Remark 1.4. In particular, if $(f_t)_{t \in \mathbb{R}}$ is a family of continues functions, non-decreasing and of degree one, such that for each x , $f_t(x)$ is non-decreasing (as a function of t), then $\rho(f_t)$ is also non-decreasing (as a function of t).

Moreover, at each irrational number the rotation number is non-decreasing:

Proposition 2. If $f_0 \prec f_1$ and if $\rho(f_0)$ is irrational, then $\rho(f_0) < \rho(f_1)$.

1.3 Flat half-critical points

We use the following notation of left and right derivative of a function f at a point p of the unit circle:

$$f'_-(p) = \lim_{x \rightarrow p, x < p} \frac{f(x) - f(p)}{x - p},$$

$$f'_+(p) = \lim_{x \rightarrow p, x > p} \frac{f(x) - f(p)}{x - p}.$$

Definition 1.5. Let p be a point of the unit circle, $n \in \mathbb{N}$, and f be a piecewise \mathcal{C}^∞ function of the circle. We say that p is half-critical of order n (for f) if the left derivative of f at p is not zero and if all the right derivatives up to order n are zero, but not the derivative of order $n + 1$. Simply, we say half-critical if it is half-critical of order 1. The point p is said flat half-critical if it is half-critical and the right derivatives of all order are zero.

In this case, if f is a function of the circle with a half-critical point p of order $k \in \mathbb{N}$, piecewise \mathcal{C}^n , \mathcal{C}^n on $\mathbb{S}^1 \setminus \{p\}$, then, by slight abuse of notation, we define $\|f\|_{\mathcal{C}^n}$:

$$\|F\|_{\mathcal{C}^n} = \sup_{\substack{x \in [0,1] \\ 1 \leq i \leq n}} \left| \frac{d^i F}{dx^i}(x) \right| + \sup_{x \in [0,1]} |F(x)|.$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is a lift of f (using the projection of \mathbb{R} onto the circle which sends \mathbb{Z} on p and satisfies $F(0) \in [0, 1)$).

Remark 1.6. We note that if $F : \mathbb{R} \rightarrow \mathbb{R}$ is j times differentiable and if $j \geq i$, then:

$$\|F\|_{\mathcal{C}^j} \geq \|F\|_{\mathcal{C}^i}$$

In other words, $\|\cdot\|_{\mathcal{C}^j}$ is increasing in j . We will use this remark in the proof of the main theorem.

1.4 Discussion and Statement of the Results

One of the main questions in the field of dynamical systems is whether a circle function is “equivalent” to a rotation. This means that if f is a continuous function defined on the circle with rotation number ρ and R_ρ is the rotation by ρ , then there exists a continuous map $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $h \circ f = R_\rho \circ h$.

If h is strictly order preserving then we say that f and R_ρ are *combinatorially equivalent* or *semi-conjugate* and if h is a homeomorphism then we say that f and R_ρ are *topologically equivalent* or *conjugate*.

Poincaré (1880) realized that any circle homeomorphism with irrational rotation number is combinatorially equivalent to a rotation. Denjoy in [3] proved that any C^1 diffeomorphism with irrational rotation number and with derivative of bounded variation is topologically equivalent to a rotation. Also in [3] Denjoy showed that the hypothesis on the derivative is essential, in fact he gave examples of C^1 diffeomorphisms with irrational rotation number which are not conjugate to a rotation. Since then examples of this kind, called Denjoy counterexamples, have been produced by many mathematicians (see [7], [6], [10]).

The aim of this paper is to construct a Denjoy counterexample with a flat interval disjoint to the wandering interval. We have been motivated to study this problem in order to understand particular flows on the two-dimensional torus called Cherry flows. In fact, the existence of such a Denjoy counterexample allows us to have some more information about the topological properties of the quasi-minimal set (for more details on the relation between circle maps with a flat interval and Cherry flows see [1], [8], [9]). For this reason we have studied the techniques used by Hall in [6] to construct a C^∞ Denjoy counterexample and in the light of the recent results in the field of circle homeomorphisms we have found the following theorem:

Theorem 1.7. *Let p be a point on the circle. For all irrational numbers $\rho \in [0, 1)$ there exists a circle homeomorphism $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with rotation number ρ which satisfies the following properties:*

- f is piecewise C^∞ ,
- f is a C^∞ diffeomorphism on $\mathbb{S}^1 \setminus \{p\}$,
- p is a flat half-critical point for f ,
- f has a wandering interval.

We notice that the homeomorphism f produced in Theorem 1.7 is a Denjoy counterexample (see Lemma 2.1 for more details).

This result, which is not what we originally hoped to prove, remains very interesting as it is the degenerate case for upper circle maps for which all remains to be explored (see [4]). The problem of the construction of a Denjoy counterexample with a flat interval remains open.

Standing Assumption. We will always denote maps on \mathbb{S}^1 with minuscule letters and the corresponding lift with the corresponding capital letter.

We will also usually abuse of the notation identifying a subset of \mathbb{S}^1 with one of its preimages under the projection $\pi : \mathbb{S}^1 \rightarrow \mathbb{R}$.

2 Some Technical Lemmas

Lemma 2.1. *Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a continuous, non-decreasing, degree one function with irrational rotation number $\rho \in [0, 1)$. Then the following statements are equivalents:*

1. f is a Denjoy counterexample,
2. f has not dense orbits,
3. f has a wandering interval, i.e. there exists a non-empty set $I \subset \mathbb{S}^1$ such that, for all $n, m \in \mathbb{N}$, $n \neq m$, $f^n(I) \cap f^m(I) = \emptyset$,
4. there exists an interval $I \subset \mathbb{S}^1$ such that $|I| > 0$ and $|f^n(I)| \rightarrow 0$ for $n \rightarrow +\infty$.

The proof of this Lemma is known and can be found for example in [6], pag. 263.

Lemma 2.2. *Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a non-decreasing, degree one and piecewise \mathcal{C}^∞ function, for which $\frac{1}{2}$ is a flat half-critical point².*

We suppose that there exists an interval I of the form $I = (\frac{1}{2}, c)$ with $c < 1$, such that: $F'(x) = 0 \Leftrightarrow x \in I$ and such that the left-sided derivative $F'_-(\frac{1}{2}) > 0$.

Then, $\forall n \in \mathbb{N}$, $\forall \delta \in (0, 1)$, and for each pair of intervals $I_1 = (\frac{1}{2}, a)$ and $J_1 = (b, c)$ such that $I_1 \cup J_1 \subset I$ and $I_1 \cap J_1 = \emptyset$, there exists a non-decreasing, degree one and piecewise \mathcal{C}^∞ function, $\tilde{f} = \tilde{f}_{n,\sigma,\delta} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, for which $\frac{1}{2}$ is flat half-critical point. Moreover \tilde{f} satisfy the following conditions:

1. $\|\tilde{F} - F\|_{\mathcal{C}^n} < \delta$,
2. $\rho(\tilde{f}) = \rho(f)$,
3. $|\tilde{F}'(x) - F'(x)| < \delta F'(x)$, $\forall x \in (0, 1) \setminus \bar{I}$,
4. $\tilde{F}'(x) = 0 \Leftrightarrow \forall x \in I_1 \cup J_1$,
5. $\tilde{F}'_-(\frac{1}{2}) > 0$.

Proof. Let $n \in \mathbb{N}$ and $0 < \delta < 1$.

We observe that $I = (\frac{1}{2}, c)$, $I_1 = (\frac{1}{2}, a)$ and $J_1 = (b, c)$, with $\frac{1}{2} < a < b < c < 1$ (because $I_1 \cap J_1 = \emptyset$). The configurations of this intervals is clarified in the following figure.

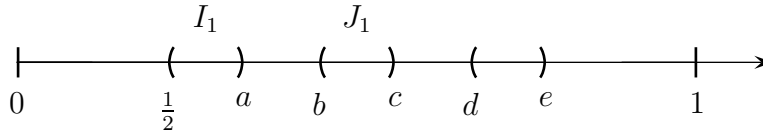


Figure 1: Positions of the points as in Lemma 2.2

Let $0 < \epsilon < \max(\frac{1}{2}, 1 - c)$, then by properties of F , the neighborhood of the flat interval $(\frac{1}{2} - \epsilon, c + \epsilon)$, has two connected components on $(0, 1)$ on which $F'(x) > \xi(\epsilon) > 0$. So, we choose $(d, e) \subset (c + \epsilon, 1)$.

We define for any $x \in [0, 1]$,

$$\tilde{F}(x) = F(x) + \frac{\delta}{3\kappa_{C_n}} \int_0^x (\phi_{a,b}(t) - \phi_{d,e}(t)) dt$$

²See Definition 1.3

where $\phi_{a,b}$ is a bump function supported on $[a, b]^3$, $\phi_{a,b} > 0$ on (a, b) . The constant $C_{d,e} > 0$ is first chosen such that $\phi_{d,e} < \xi\delta$, and then, the constant $C_{a,b} > 0$ is chosen such that:

$$\int_0^1 (\phi_{a,b}(t) - \phi_{d,e}(t)) dt = 0.$$

Finally, $\kappa = \max((b-a)C_{a,b}, (e-d)C_{d,e})$ and $c_n = 2 \max(\|\phi_{a,b}\|_{C^n}, \|\phi_{d,e}\|_{C^n}) \geq 1$.

We observe that $\tilde{F}(0) = F(0)$ and that

$$\tilde{F}(1) = F(1) + \frac{\delta}{3\kappa c_n} \int_0^1 (\phi_{a,b}(t) - \phi_{d,e}(t)) dt = F(1) + \frac{\delta}{3\kappa c_n} \times 0 = F(1) = 1$$

Moreover, \tilde{F} can be extended on \mathbb{R} (since $\tilde{F}(0) = F(0)$ and $\tilde{F}(1) = F(1)$), by:

$$\tilde{F}(x) = \tilde{F}(x - \lfloor x \rfloor) + \lfloor x \rfloor$$

(where $\lfloor x \rfloor$ denote the integer part of x), so \tilde{F} is projected on a well defined function $\tilde{f} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ which is of degree one. It is also easy to see that \tilde{f} is piecewise C^∞ and that $\frac{1}{2}$ is a flat half-critical point for \tilde{f} .

We prove that \tilde{F} satisfies the properties (1)-(5) above:

1. For $n = 0$:

$$\left| F(x) - \tilde{F}(x) \right| = \frac{\delta}{3\kappa c_n} \left| \int_0^x \phi_{a,b}(t) - \phi_{d,e}(t) dt \right| \leq \frac{\delta}{3\kappa c_n} \kappa < \delta.$$

Since the two functions $\phi_{a,b}$, $\phi_{d,e}$ have disjoint supports, then the integral is bounded by κ . For all $1 \leq k \leq n$:

$$\begin{aligned} \left| \frac{d^k F}{dx^k}(x) - \frac{d^k \tilde{F}}{dx^k}(x) \right| &= \left| \frac{\delta}{3\kappa c_n} \frac{d^{k-1}}{dx^{k-1}} \frac{d}{dx} \int_0^x \phi_{a,b}(t) - \phi_{d,e}(t) dt \right| \\ &\leq \frac{\delta}{3\kappa c_n} \sup_{x \in [0,1]} \left| \frac{d^{k-1}}{dx^{k-1}} (\phi_{a,b}(x) - \phi_{d,e}(x)) \right| \\ &\leq \frac{\delta}{3\kappa c_n} 2c_n < \delta. \end{aligned}$$

2. Since the initial perturbation \tilde{f} is C^0 small, it can be majorized by similarly small translations in either direction and for some intermediated translation the rotation number will be equal to $\rho(f)$. We observe that the property $\left\| F - \tilde{F} \right\|_{C^0} < \delta$ remains true (the translation don't change the norm of superior order) and the other properties remain also verified.

3. $\tilde{F}'(x) = F'(x) + \frac{\delta}{3\kappa c_n} (\phi_{a,b}(x) - \phi_{d,e}(x))$.

We observe that on (a, b) we have $\phi_{a,b} > 0$ and $\phi_{d,e} = 0$, so \tilde{F} is non-decreasing on (a, b) .

³The same notation is used for the interval $[d, e]$.

Moreover on (d, e) the function $\phi_{a,b} = 0$ and $\phi_{d,e} > 0$. We also have that $\phi_{d,e} < \delta\xi$ and $F' > \xi^4$; then the fact that $F' - \phi_{d,e} > 0$ implies that \tilde{F} is non-decreasing on (d, e) .

On the other intervals, we have $\tilde{F}' = F'$. This fact proves the point (3) and the fact that the function is non-decreasing. Finally, we have proved the points (3) and (4).

4. $\tilde{F}'_-(x) = F'_-(x)$ on $(0, a) \ni \frac{1}{2}$, and $F'_-(\frac{1}{2}) > 0$.

□

Lemma 2.3. *Let $\tilde{f} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a non-decreasing, degree one, piecewise \mathcal{C}^∞ function, for which $\frac{1}{2}$ is a flat half-critical point.*

We assume that \tilde{f} satisfies the following properties:

- *There exist two intervals I and J , $I = (\frac{1}{2}, a)$, $J = (a', b)$, $a' > a$, $b < 1$, such that:*

$$x \in I \cup J \Leftrightarrow \frac{d\tilde{F}}{dx}(x) = 0$$

and the left-sided derivative of \tilde{F} in $\frac{1}{2}$ is $\tilde{F}'_-(\frac{1}{2}) > 0$.

Then, $\forall n \in \mathbb{N}$, $\forall \sigma \in (0, 1)$, $\exists g = g_{n,\sigma} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a non-decreasing, degree one, piecewise \mathcal{C}^∞ function for which $\frac{1}{2}$ is a flat half-critical point. Moreover g satisfies the following conditions:

1. $\|G - \tilde{F}\|_{\mathcal{C}^n} < \sigma$,
2. $\rho(g) = \rho(\tilde{f})$,
3. $|G'(x) - \tilde{F}'(x)| < \sigma \tilde{F}'(x)$, $\forall x \in (0, 1) \setminus (\frac{1}{2}, b)$,
4. $G'(x) = 0 \Leftrightarrow x \in I$,
5. $G'_-(\frac{1}{2}) > 0$.

Proof. This proof is really similar to the proof of Lemma 2.2.

Let $n \in \mathbb{N}$ an integer, $0 < \sigma < 1$ a fixed real number and $(c, d) \subset (b, 1)$.

We have the following configuration,

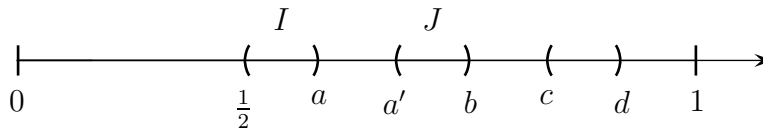


Figure 2: Positions des points du lemme 2.2

We use the bump function $\phi_{\cdot,\cdot}$ and its properties.

In this Lemma, (c, d) plays the same role as (d, e) in Lemma 2.2, the constants κ and c_n depend on (a, b) and (c, d) .

So, on (c, d) we have $\tilde{F}' > \xi$ and we choose $C_{c,d}$ such that:

$$\phi_{c,d} < \sigma\xi.$$

⁴This comes from the fact that $(d, e) \subset (c, 1)$.

This condition guarantees that the constructed function is non-decreasing (see the proof of Lemma 2.2).

As before, we choose $C_{a,b}$ in a such way that:

$$\int_0^1 (\phi_{a,b}(t) - \phi_{c,d}(t)) dt = 0.$$

We denote:

$$G(x) = \tilde{F}(x) + \frac{\sigma}{3\kappa c_n} \int_0^x (\phi_{a,b}(t) - \phi_{c,d}(t)) dt.$$

We observe that G could be extended on \mathbb{R} (because $G(0) = \tilde{F}(0)$ and $G(1) = \tilde{F}(1)$), by:

$$G(x) = G(x - \lfloor x \rfloor) + \lfloor x \rfloor$$

so it is projected on a well defined, degree one function $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ which is piecewise \mathcal{C}^∞ and for which $\frac{1}{2}$ is a flat half-critical point.

The proof of the fact that G satisfies the properties (1)-(5) is exactly the same that the proof of the Lemma 2.2 for \tilde{F} .

□

3 Proof of Theorem 1.7

Let $\rho \in [0, 1)$ be a fixed irrational number. We chose $p = \frac{1}{2}^5$, which will be the flat half-critical point.

The Denjoy counterexample which we will construct will be defined as the limit of a sequences of functions:

$$(f_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1)_{n \in \mathbb{N}},$$

for which there exist:

- an interval J_0 and a sequence of intervals:

$$(I_n)_{n \in \mathbb{N}}$$

such that

$$\bigcap_{i \geq 0} \overline{I_i} = \left\{ \frac{1}{2} \right\},$$

- a sequence of integers

$$1 = r_0 < r_1 < r_2 < \dots < r_n$$

such that, for all $i \in \{0, 1, 2, \dots, n\}$ the following conditions are satisfied:

1. f_i is piecewise \mathcal{C}^∞ , non-decreasing, degree one map,
2. $\rho(f_i) = \rho$,
3. $F'_i(x) = 0$ if and only if $x \in I_i$,

⁵Without loosing generality, the proof remains the same for any $p \in \mathbb{S}^1$, changing $\frac{1}{2}$ by p .

4. The left-sided derivative of f_i , $(F_i)'_-(\frac{1}{2}) > 0$,
5. $|I_i| < \frac{1}{2}|I_{i-1}|$,
6. $\|F_i - F_{i-1}\|_{C^{i-1}} < \frac{1}{2^i}$,
7. $0 < |f_i^j(J_0)| < \frac{1}{2^{k-1}}$, if $r_{k-1} \leq j < r_k$ for $k \in \{1, 2, \dots, i\}$,
8. $f_i^j(J_0) \cap I_i = \emptyset$, if $0 \leq j < r_i$ and $f_i^{r_i}(J_0) \subseteq I_i$,
9. $|F_i'(x) - F_{i+1}'(x)| < \frac{1}{2^{i+1}}$, for all $x \in (0, 1) \setminus \overline{I_i}$.

We prove the existence of such a sequence by induction.

3.0.1 Initialization

Construction of f_0 .

We start to construct a piecewise \mathcal{C}^∞ , non-decreasing, degree one map \tilde{f} with the following properties:

- $\tilde{F}'(x) = 0$ if and only if $x \in (\frac{1}{2}, \frac{3}{4}]$,
- The left sided derivative of \tilde{f} , $(\tilde{F})'_-(\frac{1}{2}) > 0$,

Now, let $f_0 = \tilde{f} + t(\rho)$, where t is a real number depending on ρ in the way that the rotation number of f_0 is ρ , see Remark 1.4.

We denote $I_0 = (\frac{1}{2}, \frac{3}{4}]$. Since f_0 is a diffeomorphism on $[0, 1] \setminus \overline{I_0}$ in its image, we take J_0 as any subinterval of $f_0^{-1}(I_0)$.

So f_0 is piecewise \mathcal{C}^∞ , non-decreasing, degree one map and it has rotation number ρ . The conditions (1)-(9) are also satisfied by f_0 .

3.0.2 Induction

We assume now that f_n is constructed. We construct f_{n+1} by perturbing f_n in the way that the conditions (1)-(9) are still satisfied.

We divide the interval I_n into two subintervals I_{n+1} and J_{n+1} such that:

- I_{n+1} is of the form $I_{n+1} = (\frac{1}{2}, \cdot)$,
- $I_{n+1} \cup J_{n+1} \subset I_n$,
- $I_{n+1} \cap J_{n+1} = \emptyset$,
- $f_n^{r_n}(J_0) \subset J_{n+1}$.

First step

We apply Lemma 2.2 and we get a non-decreasing, degree one and piecewise \mathcal{C}^∞ function $\tilde{f}_{n,\delta}$, which satisfies to the following conditions:

- $\|\tilde{F}_{n,\delta} - F_n\|_{\mathcal{C}^n} < \frac{\delta}{2^{n+2}},$
- $|\tilde{F}'_{n,\delta} - F'_n| < F'_n(x) \frac{\delta}{2^{n+1}}, \forall x \in (0, 1) \setminus \overline{I_n},$
- $\rho(\tilde{f}_{n,\delta}) = \rho(f_n),$
- $\tilde{F}_{n,\delta'}(x) = 0 \Leftrightarrow x \in I_{n+1} \cup J_{n+1},$
- the left-sided derivative $\tilde{F}'_{n,\delta_-}(\frac{1}{2}) > 0.$

Since, by construction, $\tilde{F}_{n,\delta}^i \rightarrow F_n^i$ for $\delta \rightarrow 0$ uniformly for $i \in \{1, 2, \dots, r_n\}$, then we can fix $\delta' < \delta < 1$, such that:

$$\left| \tilde{f}_{n,\delta'}^j(J_0) \right| < \frac{1}{2^{k-1}} \text{ for all } r_{k-1} \leq j < r_k, \quad k \in \{1, 2, \dots, n\},$$

$$\tilde{f}_{n,\delta'}^j(J_0) \cap I_n = \emptyset \text{ for all } 0 \leq j < r_n \quad (3.1)$$

and

$$\tilde{f}_{n,\delta'}^{r_n}(J_0) \subset J_{n+1}. \quad (3.2)$$

We study now the orbit of J_{n+1} under $\tilde{f}_{n,\delta'}$.

We could have two different cases:

- * there exists $m > 0$ such that $\tilde{f}_{n,\delta'}^m(J_{n+1}) \in I_n,$
- * for all $m > 0$, $\tilde{f}_{n,\delta'}^m(J_{n+1}) \notin I_n.$

Observe that the second situation never occurs. In fact, if it happens, we can approximate $\tilde{f}_{n,\delta'}$ by a \mathcal{C}^∞ , non-decreasing, degree one function which is equal to $\tilde{f}_{n,\delta'}$ everywhere except on $I_n \setminus J_{n+1}$. We have constructed a circle map with a flat interval and without dense orbits. Then by Lemma 2.1 it has a wandering interval and this is in contradiction with the corollary of Theorem 1 in [5].

We study now the first case: there exists $m > 0$ such that $\tilde{f}_{n,\delta'}^m(J_{n+1}) \in I_n.$

Since $\tilde{f}_{n,\delta'}$ can't have closed orbits ($\tilde{f}_{n,\delta'}$ has an irrational rotation number), we have:

$$\tilde{f}_{n,\delta'}^m(J_{n+1}) \in I_n \setminus J_{n+1}.$$

Moreover $\tilde{f}_{n,\delta'}^m(J_{n+1})$ depends continuously on δ , then $\tilde{f}_{n,\delta'}^m(J_{n+1})$ has a trajectory connecting $\tilde{f}_{n,\delta'}^m(J_{n+1})$ and $f_n^m(J_{n+1})$. Since $f_n^m(J_{n+1}) \notin I_n$ (f_n has an irrational rotation number), then this trajectory covers a part of I_{n+1} on the right of $\frac{1}{2}$. To conclude $\tilde{f}_{n,\delta'}^m(J_{n+1})$ is in the interior of I_{n+1} .

Second step.

By Lemma 2.3 for all $\sigma > 0$, there exists $f_{n+1,\sigma} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, piecewise \mathcal{C}^∞ , non-decreasing, degree one map such that:

- $\|F_{n+1,\sigma} - \tilde{F}_n\|_{\mathcal{C}^n} < \frac{\sigma}{2^{n+2}},$
- $|\tilde{F}'_{n+1,\sigma} - F'_n| < F'_n(x) \frac{\sigma}{2^{n+1}}, \forall x \in (0, 1) \setminus \overline{I_{n+1}},$

- $\rho(f_{n+1,\sigma}) = \rho$,
- $F'_{n+1,\sigma}(x) = 0$ if and only if $x \in I_{n+1}$,
- $(F_{n+1,\sigma})'_-(\frac{1}{2}) > 0$,

Observe that, since $f_{n+1,\sigma}^m \longrightarrow \tilde{f}_n^m$ for $\sigma \rightarrow 0$

$$f_{n+1,\sigma}^m(J_{n+1}) \Subset I_{n+1} \quad (3.3)$$

Finally we denote $r_{n+1} = r_n + m$. Since $f_{n+1,\sigma}^i \rightarrow \tilde{f}_n^i$ uniformly for all $i \in \{1, 2, \dots, r_n + m\}$ and since $\tilde{f}_n^i(J_0)$ is a singleton for $i > r_n$, then we can affirm that (we set $f_{n+1} = f_{n+1,\sigma}$):

$$|f_{n+1}^j(J_0)| < \frac{1}{2^{k-1}}$$

for all $r_{k-1} \leq j < r_k$, with $k \in \{1, 2, \dots, n\}$, and

$$|f_{n+1}^j(J_0)| < \frac{1}{2^n}$$

if $r_n \leq j < r_{n+1}$.

By (3.3), (3.1), (3.2) and since $f_{n+1,\sigma}^i \rightarrow \tilde{f}_n^i$ uniformly for all $i \in \{1, 2, \dots, r_{n+1}\}$, then

$$f_{n+1}^i(J_0) \cap I_{n+1} = \emptyset \text{ for all } 0 \leq i < r_{n+1}$$

and

$$f_{n+1}^{r_{n+1}}(J_0) \Subset I_{n+1}.$$

So we have constructed a sequence of functions $(f_n)_{n \in \mathbb{N}}$ satisfying conditions (1)-(9) for all $n \geq 1$.

We show that for $n \in \mathbb{N}$, the sequence (f_k) converges in the sense of the norm $\|\cdot\|_{\mathcal{C}^n}$ (see Remark 1.3). The sequence (f_k) is a Cauchy sequence. In fact, let $n \in \mathbb{N}$ fixed. By point (6), for all $i \in \mathbb{N}^*$,

$$\|F_i - F_{i-1}\|_{C^{i-1}} < \frac{1}{2^i}$$

So by Remark 1.6,

$$\|F_i - F_{i-1}\|_{C^n} \leq \|F_i - F_{i-1}\|_{C^{i-1}} < \frac{1}{2^i}, \text{ for all } i > n$$

where, for $i > n$, fixed, and for all $p > q > n$ we have:

$$\|F_p - F_q\|_{C^n} < \sum_{k=q-1}^{p-1} \frac{1}{2^k} < \sum_{k=q-1}^{+\infty} \frac{1}{2^k} = \frac{1}{2^{q-2}}$$

So, for all $n \in \mathbb{N}$, the sequence (f_k) converges in the sense of the norm $\|\cdot\|_{\mathcal{C}^n}$ to a circle map of class piecewise \mathcal{C}^n , which is non-decreasing, degree one map and which has rotation number ρ (see points (1) and (2)).

It is clear that the left-sided derivative in $\frac{1}{2}$ is not zero, because all the functions f_k are equal on the left of $\frac{1}{2}$, and thus the limit (point (4)).

The sequence of functions (f_k) converges uniformly to a function f , the right-sided derivatives of all orders of f_k are zero, then it also holds for the limit function f .

So, $\frac{1}{2}$ is a flat half-critical point for f .

The conditions (3) and (9) guarantee us that f has not other critical points on $\mathbb{S}^1 \setminus \{\frac{1}{2}\}$.

To conclude, by conditions (7) and (8),

$$|f_n^i(J_0)| \rightarrow 0 \text{ for } i \rightarrow +\infty$$

uniformly in n , and then

$$|f^i(J_0)| \rightarrow 0 \text{ if } i \rightarrow +\infty.$$

By Lemma 2.1 f has a wandering interval and it is not conjugated to a rotation.

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